

THE
PRIMITIVE DOUBLE MINIMAL SURFACE OF
THE SEVENTH CLASS
AND
ITS CONJUGATE.

Submitted in partial fulfillment of the requirements for the degree of Doctor of
Philosophy, in the Faculty of Pure Science, Columbia University,

BY

GRACE ANDREWS.

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THE EVENING POST JOB PRINTING HOUSE, 156 FULTON STREET.
(EVENING POST BUILDING.)

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THE PRIMITIVE DOUBLE MINIMAL SURFACE OF THE SEVENTH CLASS AND ITS CONJUGATE.

In a paper entitled "On Certain Algebraic Double Minimal Surfaces," the author, Dr. James Maclay, discusses the minimal surfaces determined by the following conditions: (1) The generating minimal curves, conveniently designated by (α) and (β) shall coincide, that is, the curve (α) shall be its own conjugate; (2) to one point of the infinitely distant circle shall correspond one tangent to the curve; (3) the curve shall have only two infinitely distant points.

These points may, without loss of generality, be taken to correspond to $u = 0$ and $u = \infty$. When this is done and the functions $F(u)$ and $f(u)$, $[f'''(u) = F(u)]$, that define the surface in connection with the equations of Weierstrass are determined so as to satisfy the prescribed conditions, they are found to take the forms

$$(1) \quad f_k(u) = \sum_1^k \frac{A_\mu}{u^\mu} - u^2 \sum_1^k \frac{A'_\mu}{(\mu-1)^\mu} + c_0 + c_1 u - c_0' u^2$$

$$(2) \quad F_k(u) = \sum_1^k \sigma_\mu \frac{A_\mu + (-1)^\mu A'_\mu u^{2\mu+2}}{u^{\mu+3}}, \quad \sigma_\mu = -\mu(\mu+1)(\mu+2)$$

On assuming the σ to be included in the A 's and putting

$$(3) \quad \mathfrak{A}_\mu = \frac{A_\mu + (-1)^\mu A'_\mu u^{2\mu+2}}{u^{\mu+3}},$$

equation (2) becomes

$$(4) \quad F_k(u) = \sum_1^k \mathfrak{A}_\mu.$$

The elements $\mathfrak{A}_\mu(u)$, $(\mu = 1, 2, \dots, k)$ composing $F_k(u)$ themselves define double surfaces. To a given value of μ correspond an infinity of surfaces arising from the variation of the arbitrary constant A_μ . These, however, are all similar since $\mathfrak{A}_\mu$ can be thrown into the form

$$(5) \quad \mathfrak{A}_\mu = c_\mu \frac{1 + (-1)^\mu u^{2\mu+2}}{u^{\mu+3}}, \quad (c_\mu \text{ real}).$$

On the other hand, except when $k = 1$ and $F_k(u)$ reduces to $\mathfrak{A}_k$ the surfaces defined by $F_k(u)$ are not all similar. Since the class of F_k and $\mathfrak{A}_k$ is the same, being expressed through the equation $C = 2\mu + 3$, F_k may appropriately be called the general; $\mathfrak{A}_k$, the primitive surface of the class $(2k + 3)$.

The present paper is a study of the primitive double surface and its conjugate corresponding to $\mu = 2$.*

I.—THE PRIMITIVE DOUBLE SURFACE, $\mu = 2$.

1. *The Equations of the Surface.*—The function defining the surface is

$$(6) \quad \mathfrak{D}_2 = \frac{1 + u^6}{u^5}, \quad (c_2 = 1)$$

or,

$$(7) \quad \Theta_2 = \frac{u^{-2} - u^4}{-24}, \quad (\Theta_2''' = \mathfrak{D}_2).$$

When Θ_2 is substituted for $f(u)$ in the second form of the equations of Weierstrass, namely

$$\begin{aligned} x &= \frac{1}{2}(1 - u^2)f''(u) + uf'(u) - f(u) + \frac{1}{2}(1 - v^2)f_1''(v) \\ &\quad + vf_1'(v) - f_1(v) \\ (8) \quad y &= \frac{i}{2}(1 + u^2)f''(u) - iuf'(u) + if(u) - \frac{i}{2}(1 + v^2)f_1''(v) \\ &\quad + ivf_1'(v) - if_1(v) \\ z &= uf''(u) - f'(u) + vf_1''(v) - f_1'(v) \end{aligned}$$

the following equations for the surface are obtained:

$$\begin{aligned} x &= \frac{1}{2} \left[-\frac{u^4 + u^{-4}}{4} + \frac{u^2 + u^{-2}}{2} \right] + \frac{1}{2} \left[-\frac{v^4 + v^{-4}}{4} \right. \\ &\quad \left. + \frac{v^2 + v^{-2}}{2} \right] = L + L_1 \\ (9) \quad y &= \frac{i}{2} \left[\frac{u^4 - u^{-4}}{4} + \frac{u^2 - u^{-2}}{2} \right] - \frac{i}{2} \left[\frac{v^4 - v^{-4}}{4} \right. \\ &\quad \left. + \frac{v^2 - v^{-2}}{2} \right] = M + M_1 \\ z &= \frac{u^3 - u^{-3}}{3} + \frac{v^3 - v^{-3}}{3} = N + N_1. \end{aligned}$$

The paper already cited determines the plane lines of curvature and the straight lines of the general surface $F_k(u)$, the properties of the minimal curve and the infinitely distant elements of the primitive surface; the nature of the geodesic in the xy -plane by which the primitive surfaces μ even are characterized and the remaining curves of intersection in the same plane.

2. *Plane Lines of Curvature and Straight Lines of the Surface.*—When a surface is represented upon the unit sphere

*The double surface corresponding to $\mu = 1$ has been treated by Schilling in the Inaugural Dissertation, "Die Minimalflächen fünfter Klasse."

through parallel normals, plane lines of curvature pass into small circles with planes parallel to the lines of curvature and right line asymptotes into great circles with planes perpendicular to the right lines. It is therefore possible by forming the equations of the stereographic projections of these circles on the plane $Z = 0$ and combining them with the differential equations of the plane lines of curvature and straight lines to determine the latter.

When this is done, it is found that no non-meridian circles represent curves of the desired type, except the pole $uv = 0$, which as it involves $u = 0$ corresponds to a line at infinity, and the equatorial circle of the sphere $uv - 1 = 0$. The relation $uv - 1 = 0$ determines upon the surfaces μ even, a line of curvature in the xy -plane and upon the surfaces μ odd, the z -axis.

It is found that meridian circles cannot represent curves of the desired types unless the arbitrary constants A_μ involved in $F_k(u)$ fulfill certain conditions. When every A_μ but one is zero, F_k reduces to $\mathfrak{P}_k$. In this case, the surface possesses $(\mu + 1)$ lines of curvature lying in planes that pass through the z -axis, and $(\mu + 1)$ straight lines lying in the xy -plane and passing through the origin. When μ is even, the planes, among which is found the xz -plane, pass through the lines, which include the x -axis.

As all the plane lines of curvature on the surface are represented on the sphere in great circles, they are geodesics, and by a theorem due to Schwarz determine planes of symmetry. By a second theorem, the straight lines are axes of symmetry.

It follows from these results that the surface $\mu = 2$ has four planes of symmetry, including the xy and xz -planes and three axes of symmetry, including the x -axis.

3. *The Minimal Curve.*—The order, class and rank of the minimal curve of the primitive surfaces are found to be $2\mu + 4$, $2\mu + 2$, $2\mu + 4$, respectively.

The stationary points of the curve are furnished by the values of u satisfying.

$$(10) \quad \mathfrak{D}_\mu(u) = 0; \text{ or, } u^{2\mu+2} = (-1)^{\mu+1},$$

When μ is even, a pair of conjugate points corresponding to u and $-u$ are found to be symmetrically disposed with respect to the xy -plane. The middle point of the line joining a pair lies on the surface.

The plane of osculation at the infinitely distant points of

the curve falls into the plane at infinity and the tangent line becomes the tangent to the imaginary circle.

When $\mu = 2$, the order, class and rank are 8, 6 and 8 respectively, and equation (10) takes the form

$$(11) \quad u^6 = -1.$$

4. *The Infinitely Distant Elements.*—These, as is known, lie on straight lines obtained by joining the infinitely distant points of one minimal curve to the infinitely distant points of the other. They arise therefore from the four combinations (1) $u = 0, v = 0$; (2) $u = \infty, v = \infty$; (3) $u = 0, v = \infty$; (4) $u = \infty, v = 0$. When these values are introduced into the equations of the primitive surfaces

$$(12) \quad \begin{aligned} x + iy &= \frac{(-1)^{\mu+1} u^{\mu+2}}{\mu+2} + \frac{u^{-\mu}}{\mu} - \frac{v^{-\mu-2}}{\mu+2} + \frac{(-1)^{\mu} v^{\mu}}{\mu} \\ x - iy &= \frac{-u^{-\mu-2}}{\mu+2} + \frac{(-1)^{\mu} u^{\mu}}{\mu} + \frac{(-1)^{\mu+1} v^{\mu+2}}{\mu+2} + \frac{v^{-\mu}}{\mu} \\ z &= \frac{(-1)^{\mu} u^{\mu+1} - u^{-\mu-1}}{\mu+1} + \frac{(-1)^{\mu} v^{\mu+1} - v^{-\mu-1}}{\mu+1} \end{aligned}$$

it is found that combinations (1) and (2) give rise to the line

$$(13) \quad z = 0, \quad \tau = 0,$$

and (3) and (4) to the lines

$$(14) \quad x + iy = 0, \quad \tau = 0,$$

$$(15) \quad x - iy = 0, \quad \tau = 0,$$

respectively. $\tau = 0$ is the equation of the plane at infinity.

The lines are not simple lines of the surface. This may be shown in the case of the first line as follows: assume $u = \infty, v = \infty$, and regard only the terms of the highest degree in equations (12). We have then

$$(16) \quad x + iy = \frac{(-1)^{\mu+1} u^{\mu+2}}{\mu+2}, \quad x - iy = \frac{(-1)^{\mu+1} v^{\mu+2}}{\mu+2},$$

from which we derive

$$(17) \quad \frac{x + iy}{x - iy} = \left(\frac{u}{v} \right)^{\mu+2}$$

To every point of the straight line correspond $(\mu+2)$ values of the ratio $\frac{u}{v}$, when $u = \infty, v = \infty$.

But now assuming $u = 0, v = 0$, and regarding only the terms of lowest degree, we have

$$(18) \quad x + iy = -\frac{v^{-\mu-2}}{\mu+2}, \quad x - iy = -\frac{u^{-\mu-2}}{\mu+2}$$

and therefore

$$(19) \quad \frac{x + iy}{x - iy} = \frac{v^{-\mu-2}}{u^{-\mu-2}} = \left(\frac{u}{v}\right)^{\mu+2}$$

To every point of the straight line correspond $(\mu + 2)$ values of the ratio $\frac{u}{v}$, when $u = 0, v = 0$. Apparently the line is of multiplicity $2(\mu + 2)$. But for a given point, to every pair of values u, v satisfying (16) and (17) correspond a pair $-v^{-1}, -u^{-1}$ satisfying (18) and (19). It is characteristic of double surfaces that every point corresponds to two pairs of values of the parameters, namely, u, v and $-v^{-1}, -u^{-1}$ therefore the multiplicity of the line under consideration is $(\mu + 2)$.

The two imaginary lines (14) and (15) are also of multiplicity $(\mu + 2)$.

5. *The Geodesic in the xy -Plane.*—By employing the equations of Schwarz defining a minimal surface in terms of a curve through which the surface passes, the following relation is obtained between ρ the radius of curvature of the geodesic and the parameter u

$$(20) \quad \rho = u^2 \mathfrak{D}_\mu(u).$$

Through the substitutions $u = e^{i\phi}$ and $\varphi = \frac{\pi}{2} - \frac{\psi}{\mu}$, (20)

may be changed into

$$(21) \quad \rho = 4 \sin \frac{\mu+1}{\mu} \psi.$$

In this form ρ is recognized as the radius of curvature of a hypocycloid generated by means of a fixed and a rolling circle with radii equal respectively to

$$R = \frac{4(\mu+1)}{\mu(\mu+2)} \text{ and } r = \frac{2}{\mu+2}$$

ψ is the angle between the line of centers of the two circles and the x -axis.

From the relation between R and r , it is seen that the curve has $(\mu + 1)$ cusps, one of which lies on the x -axis. Since a cusp occurs wherever $\rho = 0$, a comparison of (20) and (10) shows that the points in the xy -plane furnished by pairs of conjugate stationary points of the minimal curve are the cusps of the hypocycloid.

6. *The Curves of Intersection in the xy -Plane.*—It is of advantage to determine these for the surface $\mu = 2$ by a special discussion and to reserve until later a statement of the more general results reached for the surfaces μ even.

When for u and v are substituted $\lambda \varepsilon^{i\phi}$ and $\lambda \varepsilon^{-i\phi}$ equation (9) takes the form.

$$\begin{aligned} x &= -\frac{1}{4}(\lambda^4 + \lambda^{-4}) \cos 4\phi + \frac{1}{2}(\lambda^2 + \lambda^{-2}) \cos 2\phi \\ (22) \quad y &= -\frac{1}{4}(\lambda^4 + \lambda^{-4}) \sin 4\phi - \frac{1}{2}(\lambda^2 + \lambda^{-2}) \sin 2\phi \\ z &= \frac{2}{3}(\lambda^3 - \lambda^{-3}) \cos 3\phi. \end{aligned}$$

The substitution $\rho = \lambda - \lambda^{-1}$ obviates the difficulties that arise from the fact that each point of a double surface corresponds to one value of the parameter ϕ , combined with two values of the parameter λ , namely λ and $-\lambda^{-1}$. By this substitution equations (22) become

$$\begin{aligned} x &= -\frac{1}{4}(\rho^4 + 4\rho + 2) \cos 4\phi + \frac{1}{2}(\rho^2 + 2) \cos 2\phi \\ (23) \quad y &= -\frac{1}{4}(\rho^4 + 4\rho^2 + 2) \sin 4\phi - \frac{1}{2}(\rho^2 + 2) \sin 2\phi \\ z &= \frac{2}{3}\rho(\rho^2 + 3) \cos 3\phi. \end{aligned}$$

The curves of intersection in the xy -plane are determined by the conditions

$$(A) \rho = 0, \quad (B) \rho^2 + 3 = 0, \quad (C) \cos 3\phi = 0.$$

Condition (A) leads to the equations

$$\begin{aligned} (24) \quad x &= -(\cos \psi + \frac{1}{2} \cos 2\psi) \\ y &= -(\sin \psi - \frac{1}{2} \sin 2\psi). \end{aligned}$$

where $\psi = \pi - 2\phi$. These define a hypocycloid generated by a fixed and a rolling circle of radii $3/2$ and $1/2$ respectively. The radius of curvature ρ' satisfies the equation

$$(25) \quad \rho' = 4 \sin 3/2 \psi = 4 \cos 3\phi.$$

The curve therefore has three cusps, one of which is on the x -axis at $x = -3/2$. The corresponding vertex is at $x = 1/2$.

Since $\lambda = 1$ and $uv - 1 = 0$, when $\rho = 0$, the curve is the geodesic in the xy -plane already mentioned.

Its degree may be determined by putting $v = u^{-1}$ in equations (9) and substituting the resulting values of x and y in the equations of an arbitrary line. This gives an equation of the fourth degree in u^2 . The degree of the curve is four. The multiplicity is apparently two, but is really one, since the two pairs of values of the parameters to which each point of a double surface corresponds, namely u, v and $-v^{-1}, -u^{-1}$ become for points on the curve in question u, v and $-u, -v$.

Condition (B) leads to the equations

$$(26) \quad \begin{aligned} x &= \frac{1}{2} (\cos \psi + \frac{1}{2} \cos 2\psi) \\ y &= \frac{1}{2} (\sin \psi - \frac{1}{2} \sin 2\psi). \end{aligned}$$

These evidently define a hypocycloid derived from the preceding by means of the proportional factor $-\frac{1}{2}$. The curve has a cusp at $x = 3/4$, $y = 0$ and a vertex at $x = -1/4$, $y = 0$. It is double since each point corresponds to two values of ρ and is isolated since these values are imaginary; for an infinitesimal change in ρ makes the co-ordinates of the surface imaginary.

Condition (C) implies the relations

$$(27) \quad \begin{aligned} \sin 6\phi &\equiv \sin 4\phi \cos 2\phi + \cos 4\phi \sin 2\phi = 0 \\ \frac{\sin 4\phi}{\sin 2\phi} &= -\frac{\cos 4\phi}{\cos 2\phi} = 1. \end{aligned}$$

The equations of the section therefore become

$$(28) \quad \begin{aligned} \frac{x}{\cos 2\phi} &= -\frac{y}{\sin 2\phi} = \frac{1}{4} (\rho^4 + 6\rho^2 + 6) \\ z &= 0. \end{aligned}$$

These are the equations of three straight lines of multiplicity 4, namely :

$$y = 0, \quad y = \pm \sqrt{3} x.$$

Regarded as an equation in ρ^2 , (28) is satisfied by

$$(29) \quad \rho^2 = -3 \pm \sqrt{3 + \frac{4x}{\cos 2\phi}}$$

Two of the four values of ρ are real if $\frac{x}{\cos 2\phi} > 3/2$. Otherwise all are imaginary. Comparison with equations (24) and (25) shows that the projections of the cusps of the geodesic hypocycloid upon the x -axis are given by

$$x = 3/2 \cos 2\phi, \text{ when } \cos 3\phi = 0.$$

The straight lines therefore proceed outward from the cusps of the hypocycloid to infinity in connection with real parts of the surface. Two real nappes pass through each.

7. *Order of the Surface.*—On adding together the degrees of the various curves of intersection in the xy -plane, regard being paid to the multiplicity of each, the order of the surface is found to be 28. This accords with the result obtained from the formula

$$O = \frac{1}{2} m (m - 1),$$

when m is the order of the minimal curve.

There are in the $x y$ -plane

3	finite straight lines of multiplicity 4.....	12
1	straight line at infinity of multiplicity 4.....	4
1	hypocycloid of multiplicity 1, degree 4.....	4
1	" " " " 2, " 4.....	8
		28

8. *Curves of Intersection in the xz -Plane.*—These are determined by the conditions

$$(A) \cos \varphi = 0, \quad (B) \sin \varphi = 0, \quad (C) \cos 2\varphi = -\frac{\lambda^2 (\lambda^4 + 1)}{\lambda^8 + 1}$$

$$= -\frac{\rho^2 + 2}{\rho^4 + 4\rho^2 + 2}.$$

(*A*) is the condition for the x -axis.

(*B*) leads to a curve of the fourth degree with the equations

$$(30) \quad \begin{aligned} x &= -\frac{1}{4}(\rho^4 + 2\rho^2 - 2), \\ y &= \frac{2}{3}\rho(\rho^2 + 3). \end{aligned}$$

The values of the direction cosines of the normal to the surface at a point u, v are

$$(31) \quad X = \frac{u+v}{uv+1}, \quad Y = i \frac{v-u}{uv+1}, \quad Z = \frac{uv-1}{uv+1}.$$

Since $u = v$ when $\sin \varphi = 0$, the normals to the surface along the curve defined by (30) fall into the xz -plane. The curve is therefore a geodesic.

The slope of the tangent at any point is derived from

$$dx = -\rho(\rho^2 + 1) d\rho, \quad dz = 2(\rho^2 + 1) d\rho.$$

$$\frac{dz}{dx} = -2/\rho.$$

The substitution of $-\rho$ for ρ shows that the curve is symmetric with respect to the x -axis which it cuts orthogonally at $x = 1/2$ and in an isolated double point at $x = -1/4$. In addition to this singularity, it has two imaginary cusps corresponding to the roots of $\rho^2 + 1 = 0$. Since it is unicursal and of degree 4, it can have no other double points.

It cuts the z -axis in four points, two real at $z = \pm 2.128$ and two imaginary. It proceeds to infinity without maxima, minima or points of inflection and becomes parallel at infinity to the x -axis. It is of the type of the ordinary parabola.

Condition (C) gives rise to a double curve of the tenth degree with the equations

$$(32) \quad \begin{aligned} x &= \frac{1}{4} \frac{(\rho^4 + 6\rho^2 + 6)(\rho^4 + 2\rho^2 - 2)}{\rho^4 + 4\rho^2 + 2}, \\ z &= -\frac{\sqrt{2}}{3} \rho^2 (\rho^4 + 6\rho^2 + 6) \left(\frac{\rho^2 + 3}{\rho^4 + 4\rho^2 + 2} \right)^{3/2} \end{aligned}$$

The curve cuts the x -axis at the origin, an isolated point, and at $x = -3/2$ and $x = 3/4$, cusps of the hypocycloids in the xy -plane.

Since to one value of x correspond four values of ρ^2 , the curve has eight branches arranged in pairs symmetric to the x -axis. On throwing the equation for x into the form of a biquadratic in τ , ($\tau = \rho^2$) and forming the discriminant

$$\Delta = 64/9 (18x^5 + 82x^4 + 216x^3 + 306x^2 + 270x + 81)$$

it is found that the only real value of x for which Δ is zero is one that lies between $-.5$ and $-.6$ and may be conveniently designated by a_0 . For $x < a_0$, two values of τ are imaginary; for $x > a_0$, all are real. For $-3/2 \leq x \leq \infty$, one of the real roots is positive. This corresponds to the non-isolated part of the curve.

When with these results are combined the facts presented in the following table, showing the comparative values of τ , x and z

τ	$-\infty$	$\dots$	-4.7	$\dots$	-3.4	$\dots\dots$	-3	$\dots$	-2.7	$\dots$	-1.2	$\dots$	$-.58$	$\dots\dots$	0	$\dots$	$.73$	$\dots$	∞
x	∞	$+$	0	$+$	$-\infty$	$+$	0	$+$	0	$+$	0	$+$	∞	$+$	0	$+$	0	$+$	∞
z		Imaginary.					Real.		Imaginary					Real.		Real.			

the conclusion is reached that the curve, so far as it is real, consists of two detached parts; the first wholly isolated, associated with values of τ ranging from -3.4 to -3 ; the second partly isolated, associated with values ranging from $-.58$ to ∞ .

Further light is shed upon the course of the curve through the values of dx and dz .

$$dx = \frac{1}{2} \frac{\tau^5 + 10\tau^4 + 36\tau^3 + 56\tau^2 + 44\tau + 24}{(\tau^2 + 4\tau + 2)^2} d\tau$$

$$dz = \frac{1}{\sqrt{2}} \left(\frac{\tau + 3}{\tau^2 + 4\tau + 2} \right)^{1/2} \frac{\tau^5 + 10\tau^4 + 36\tau^3 + 56\tau^2 + 44\tau + 24}{(\tau^2 + 4\tau + 2)^2} d\tau$$

$$\frac{dz}{dx} = \sqrt{2} \left(\frac{\tau + 3}{\tau^2 + 4\tau + 2} \right)^{1/2}$$

Since the equation

$$\tau^5 + 10\tau^4 + 36\tau^3 + 56\tau^2 + 44\tau + 24 = 0$$

has but one real root and that lies between 0 and -3 ; and the equation

$$\tau^2 + 4\tau + 2 = 0$$

has the roots -3.4 and $-.58$, the common factors of dx and dz determine no cusps upon the real branches of the curve.

The wholly isolated part of the curve resembles a semi-cubic parabola with a cusp on the x -axis at $x = 3/4$, from which point the curve extends in the positive direction to infinity. The second part consists of two branches parallel to the z -axis at $x = \infty$, $z = \pm \infty$ when they take their rise, cutting each other at an angle of 120° on the x -axis at $x = -3/2$, proceeding from this point in connection with real parts of the surface, intersecting the geodesic in the two points ± 2.128 on the z -axis and becoming parallel to the x -axis at infinity.

To recapitulate, the curves in the xz -plane are:

1 straight line (the x -axis) of multiplicity 4.....	4
1 geodesic of degree 4, and multiplicity 1.....	4
1 curve of degree 10, and multiplicity 2.....	20
	28

9. *Symmetry of the Surface.*—It is unnecessary to make a special study of the curves of intersection in the planes $y = \pm \sqrt{3}x$. The sections of a surface by a set of planes of symmetry that pass through a straight line, as the z -axis, and are so related that the angles between consecutive planes are equal, cannot be of more than two types; for any one of the planes (1), (3), (5)..... is symmetric to some other of the same group with respect to some one of the planes (2), (4), (6)..... and *vice versa*. If n , the number of planes, is odd, the n^{th} plane belongs to the same group as the first, but at the same time is symmetric to the second plane with respect to the first, so that the two types of section furnished by consecutive planes can differ only in the position of their component curves with respect to the origin.

In the present case since the curves in the xz -plane are symmetric to the x -axis, the two types of section made by the three vertical planes differ only in the relation of their curves to the z -axis. The geodesics and double curves in all three planes pass through the same two points on the z -axis. These are singular points for the surface of multiplicity three.

The determination of the planes and axes of symmetry of a minimal surface by means of its plane geodesics and straight lines is not necessarily exhaustive since the theorems converse to those of Schwarz are not true. But that in the pres-

ent instance it is complete is readily seen. If the double surface $\mu = 2$ should possess an additional plane of symmetry, the latter must bisect the angle between two of the planes $y = 0$, $y = \pm \sqrt{3} x$ since in any other position it would require the existence of a second set of straight lines lying on the surface. The existence of one such plane would imply the existence of three bisecting the angles between the planes of symmetry already found, and presenting a third type of section. As no such type can exist, neither can the assumed planes.

There can be no additional axes of symmetry; for they could be no other than a set of three bisecting the angles between the straight lines of the surface, and in this position since the surface is symmetrical to the xy -plane, they would with the z -axis determine three additional planes of symmetry.

It is possible to place some limitation on the nature of the curves of intersection of an algebraic surface and a plane of symmetry. Such curves, if neither imaginary nor multiple, must be geodesics. If $F(x, y, z) = 0$ represents an algebraic surface referred to a plane of symmetry as the xy -plane, $F(x, y, z)$ contains only even powers of z and $\partial F / \partial z$ contains odd powers in every term. The direction cosines of the normal at (x, y, z) are :

$$\frac{\frac{\partial F}{\partial x}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}, \quad \frac{\frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}},$$

$$\frac{\frac{\partial F}{\partial z}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2}}.$$

Along a curve of section in the xy -plane, $\frac{\partial F}{\partial z} = 0$. Un-

less at the same time $\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2} = 0$, that is, unless the curve is multiple, the normal to the surface falls into the xy -plane and coincides with the normal to the curve. The latter is therefore a geodesic.

10. *Curves Corresponding to the Parallels and Meridians on the Sphere.*—These constitute two systems cutting each other orthogonally and obtained respectively by means of the relations

$$(33) \quad uv = \lambda^2 = c,$$

and

$$(34) \quad v = \beta u \text{ or } \phi = k.$$

The λ Curves.—The λ curves, as they may be called, are of the eighth degree, with the exception of two curves of the fourth degree, namely, the geodesic hypocycloid, $uv = 1$, and the real line at infinity, $uv = 0$.

Since $\lambda \varepsilon^i \phi = -\lambda \varepsilon^i (\pi + \phi)$, $\lambda \varepsilon^{-i} \phi = -\lambda \varepsilon^{-i} (\pi + \phi)$, and on a double surface $\lambda \varepsilon^i \phi$, $\lambda \varepsilon^{-i} \phi$ and $\frac{1}{\lambda} \varepsilon^i (\pi + \phi)$, $\frac{1}{\lambda} \varepsilon^{-i} (\pi + \phi)$ furnish the same point, all of the curves of the system are obtained by assigning to λ values ranging from 1 to ∞ . λ being fixed, the substitutions φ , $\pi - \varphi$, $\pi + \varphi$ show the curves to be symmetric with respect to the xy - and xz -planes and to the x -axis, and therefore to all the planes and axes of symmetry.

The curves are closed. With the exception of the geodesic hypocycloid, each curve cuts the xy -plane in three double points, one on each of the straight lines of the surface and cuts each vertical plane of symmetry in a double point on the straight line, two ordinary points on the geodesic and two double points on the double curve. One curve of the system cuts the z -axis in the two singular points. No two of the curves intersect.

The φ Curves.—These are of the fourth degree. They include the vertical geodesics and the straight lines of the surface. All the curves of the system are obtained by assigning to φ values ranging from 0 to π , and all the points of a curve, by permitting λ to vary from 0 to ∞ .

Each curve is symmetric with respect to the xy -plane and to φ and $\pi - \varphi$ correspond curves symmetrically situated with respect to the xz -plane and intersecting upon the double curve in that plane if $45^\circ \leq \varphi \leq 90^\circ$.

From the direction cosines of the tangent

$$\frac{-(\lambda^8 - 1) \cos 4\varphi + \lambda^2 (\lambda^4 - 1) \cos 2\varphi}{P^{1/2}},$$

$$\frac{-(\lambda^8 - 1) \sin 4\varphi - \lambda^2 (\lambda^4 - 1) \sin 2\varphi}{P^{1/2}}, \quad \frac{2(\lambda^6 + 1) \cos 3\varphi}{P^{1/2}},$$

where

$$P^{1/2} =$$

$$\sqrt{(\lambda^8 - 1)^2 + (\lambda^6 - \lambda^2)^2 - 2(\lambda^8 - 1)(\lambda^6 - \lambda^2) \cos 6\varphi + 4\lambda^2 (\lambda^6 + 1)^2 \cos^2 3\varphi}$$

it is evident that the curves cut the geodesic hypocycloid orthogonally except at the cusps and become parallel to the xy -plane at infinity.

11. *Model of the Surface.*—By making a framework of cardboard representing the non-isolated curves of intersection in the planes of symmetry, and using this to support a network of wires representing certain φ and λ curves, I constructed a skeleton model of the surface. The following values, multiplied throughout by two, constitute the data for the construction.

GEODESIC HYPOCYCLOID.			GEODESIC IN xz -PLANE.			DOUBLE CURVE IN xz -PLANE.		
ψ	x	y	ρ	x	z	$\tau (= \rho^2)$	x	z
0°	—1.50	0	0	.50	0	0	—1.5	0
20°	—1.32	— .02	.1	.49	.20	.1	—1.23	.45
40°	— .85	— .15	.2	.48	.40	.2	— .99	.82
60°	— .25	— .43	.3	.45	.62	.3	— .79	1.12
80°	.30	— .81	.4	.41	.84	.4	— .59	1.39
90°	.50	—1.00	.5	.36	1.08	.5	— .40	1.63
110°	.73	—1.26	.6	.29	1.34	.6	— .23	1.85
120°	.75	—1.30	.7	.19	1.63	.7	— .04	2.06
130°	.73	—1.26	.8	.08	1.94	.8	.12	2.27
150°	.62	— .93	.86	0	2.13	.9	.29	2.46
170°	.51	— .34	.9	— .07	2.29	1.0	.46	2.65
180°	.50	0	1.0	— .25	2.70	1.1	.64	2.83
.....			1.686	— .46	3.09	1.2	.82	3.01

The parallels and meridians of the sphere are conformally represented on the plane $Z = 0$ by concentric circles around the origin and their radii. These correspond respectively to $\lambda^2 = C$ and $\phi = k$. They in turn through the logarithmic function are conformally represented on a second plane by lines parallel to the axes with equations

$$x = \log \lambda \text{ and } y = \varphi.$$

The λ and φ curves chosen for the construction of the model correspond to lines at intervals of $\frac{\pi}{24}$. The following table presents the points of intersection of the two sets of curves. On account of the symmetry of the surface it is unnecessary to give $\log \lambda$ and φ values greater than $4\pi/24$.

The curve $\lambda = 1.515$ is introduced because it is the λ -curve that passes through the multiple points on the z -axis.

λ		$\varphi = 0$	$\varphi = \frac{\pi}{24}$	$\varphi = \frac{2\pi}{24}$	$\varphi = \frac{3\pi}{24}$	$= \varphi \frac{4\pi}{24}$
$\varepsilon^0 = 1$	x	.5	.53	.62	.71	.75
	y	0	— .50	—0.94	—1.21	—1.30
	z	0	0	0	0	0
$\frac{\pi}{\varepsilon^{24}} = 1.14$	x	.46	.50	.61	.73	.80
	y	0	— .55	—1.01	—1.30	—1.39
	z	.54	.49	.38	.20	0
$\frac{2\pi}{\varepsilon^{24}} = 1.299$	x	.34	.38	.59	.81	.97
	y	0	— .69	—1.26	—1.61	—1.68
	z	1.16	1.06	.81	.44	0
$\frac{3\pi}{\varepsilon^{24}} = 1.48$	x	.05	.18	.51	.94	1.29
	y	0	— .98	—1.76	—2.21	—2.25
	z	1.96	1.81	1.39	.75	0
$\lambda = 1.515$	x	0	.14	.50	.97	1.37
	y	0	—1.04	—1.86	—2.33	—2.37
	z	2.13	1.97	1.51	.81	0
$\frac{4\pi}{\varepsilon^{24}} = 1.686$	x	— .46	— .24	.36	1.13	1.83
	y	0	—1.44	—2.58	—3.19	—3.16
	z	3.07	2.83	2.12	1.17	0

I am indebted to the kindness of Professor Hallock of Columbia University for the model of which two views are presented in Plate I. Professor Hallock formed three shells of paper upon a plaster impression of a clay model that I had made of the large concavity shown in Fig. 1. By joining these shells and supplying minor parts above and below the singular points on the axis of z , he completed the model depicted.

II. THE CONJUGATE SURFACE $\mu = 2$.

12. *General Relations Between a Minimal Surface and Its Conjugate.*—By virtue of these, many of the properties of the surface now to be treated can be inferred at once from those of the surface just discussed.

As the equations of the conjugate are derived from those of the original surface by the substitution of $iF(u)$, $-iF_1(v)$ for $F(u)$, $F_1(v)$ or of $if(u)$, $-if_1(v)$ for $f(u)$, $f_1(v)$, the conjugate of a double surface cannot itself be double; for if $F(u)$ satisfies the functional relation for double surfaces, $iF(u)$ does not.

Corresponding curves on a surface and its conjugate, that is, curves arising from the same relation between u and v ,

have the same spherical image. Corresponding elements are orthogonal. Lines of curvature on the one surface correspond to asymptotic lines on the other and *vice versa*.

The origin is a conical point on the conjugate of a double surface. It is also a point of symmetry. From the characteristic property already noted that each point corresponds to two sets of values of the parameters, u, v and $-v^{-1}, -u^{-1}$, it follows that the co-ordinates of a double surface may be represented by equations

$$\begin{aligned} (35) \quad x &= \varphi(u) + \varphi_1(v) &= \varphi(-v^{-1}) + \varphi_1(-u^{-1}) \\ y &= \sigma(u) + \sigma_1(v) &= \sigma(-v^{-1}) + \sigma_1(-u^{-1}) \\ z &= \chi(u) + \chi_1(v) &= \chi(-v^{-1}) + \chi_1(-u^{-1}) \end{aligned}$$

Evidently $\varphi(u) = \varphi_1(-u^{-1})$, $\varphi_1(v) = \varphi(-v^{-1})$ and similar relations connect σ, σ_1 and χ, χ_1 so that the coordinates of the conjugate surface become

$$\begin{aligned} (36) \quad x &= i[\varphi(u) - \varphi_1(v)] &= -i[\varphi(-v^{-1}) - \varphi_1(-u^{-1})] \\ y &= i[\sigma(u) - \sigma_1(v)] &= -i[\sigma(-v^{-1}) - \sigma_1(-u^{-1})] \\ z &= i[\chi(u) - \chi_1(v)] &= -i[\chi(-v^{-1}) - \chi_1(-u^{-1})] \end{aligned}$$

These equations show that to each point of the double surface correspond on the conjugate two points symmetrically situated with respect to the origin.

To one curve upon a double surface may correspond two upon the conjugate. The values u, v and $-v^{-1}, -u^{-1}$ determine upon the unit sphere points (x, y, z) , $(-x, -y, -z)$. The complete spherical image of a curve on the double surface must therefore be symmetric with respect to the origin, but it may consist of one curve, as for instance a meridian, the two halves of which are given by the first and second sets of values of the parameter respectively, or of two equal independent symmetrically situated curves, as, for instance, two parallels. In the first case, the spherical image will represent a single curve on the conjugate surface; in the second case, two.

Since the two generating minimal curves of a conjugate surface are derived from those of the original by a transformation that simply multiplies the distance of each point from the origin by a constant factor, i in the case of one curve, $-i$ in that of the other, it follows that finite points remain finite; infinitely distant points remain infinitely distant; the multiplicity of points is unchanged; stationary points correspond to stationary points; coincident points of the two curves at infinity are unaltered, as are also the order, class

and rank of the curves, the multiplicity of the imaginary circle at infinity upon their tangential surfaces and the relation of the plane of osculation and the tangent at infinitely distant points to the infinitely distant plane and circle.

As it is upon these properties that the determination of the infinitely distant elements and of the class and order of a minimal surface depends, it follows that a surface and its conjugate must have the same infinitely distant elements, class and order unless the original surface is double. In this case the class and order of the latter will be and the multiplicity of the lines at infinity may be, only one-half of the corresponding values for the conjugate surface. This is due to reductions of the type already encountered, resulting from the double generation of the double surface.

13. *Equations of the Surface.*—From these general considerations we turn to the discussion of the surface conjugate to the primitive double surface of the seventh class. Obviously it is related to the surface $iF_2(u)$ as the surface $\mathfrak{S}_2(u)$ is related to that defined by $F_2(u)$. It may be called the primitive conjugate surface $\mu = 2$.

Its equations in a form corresponding to (9) are

$$\begin{aligned} x &= i(L - L_1) \\ (37) \quad y &= i(M - M_1) \\ z &= i(N - N_1). \end{aligned}$$

and in that corresponding to (22) are

$$\begin{aligned} x &= \frac{1}{4}(\lambda^4 - \lambda^{-4}) \sin 4\varphi - \frac{1}{2}(\lambda^2 - \lambda^{-2}) \sin 2\varphi, \\ (38) \quad y &= -\frac{1}{4}(\lambda^4 - \lambda^{-4}) \cos 4\varphi - \frac{1}{2}(\lambda^2 - \lambda^{-2}) \cos 2\varphi, \\ z &= -\frac{2}{3}(\lambda^3 + \lambda^{-3}) \sin 3\varphi. \end{aligned}$$

14. *Plane Lines of Curvature and Straight Lines of the Surface.*—To the geodesic in the xy -plane of the double surface must correspond an asymptotic line on the conjugate. This must be a straight line perpendicular to the xy -plane since its spherical image is a great circle parallel to that plane. Also since every point of the geodesic is represented on the straight line by two points symmetric with respect to the origin, the line must pass through the origin. It is therefore the axis of z .

Similarly to the geodesics in the planes $y = 0, y = \pm \sqrt{\frac{1}{3}}x$ correspond three straight lines in the xy -plane, $x = 0, x = \mp \sqrt{\frac{1}{3}}y$. To the straight lines $y = 0, z = 0; y = \pm \sqrt{\frac{1}{3}}x, z = 0$ correspond three geodesics in the planes $x = 0, x = \mp \sqrt{\frac{1}{3}}y$. The

conjugate surface is therefore characterized by four straight lines, among which are the y - and z -axes, and three plane geodesics, one of which lies in the yz -plane.

The surface can have no other straight lines, nor plane geodesics, nor any plane lines of curvature other than geodesics since these would imply the existence upon the double surface of plane geodesics or straight lines additional to those already found or of asymptotic lines represented on the sphere by small circles.

15. *The Minimal Curves.*—The properties of these require no special discussion, with the exception of the relation of conjugate pairs of stationary points to the surface.

Such points are obtained by putting conjugate values of u and v derived from

$$u^6 = -1 \text{ and } v^6 = -1$$

into the equations of the minimal curves (α') and (β') . These become for a pair of values, u and u^{-1} .

$$(\alpha') \quad x = 2iL, \quad y = 2iM, \quad z = 2iN,$$

$$(\beta') \quad x = -2iL_1, \quad y = -2iM_1, \quad z = -2iN_1,$$

$$= -2iL, \quad = -2iM, \quad = 2iN.$$

The conjugate points as was to be anticipated are symmetrically disposed to the axis of z .

Equations (37) when u and u^{-1} are put for u and v become

$$(39) \quad \begin{aligned} x = y = 0 \\ z = \frac{2i}{3} \frac{u^6 - 1}{u^3} \end{aligned}$$

Under the condition $u^6 = -1$, this value of z becomes $\pm 4/3$. The pairs of points under discussion are therefore connected with two points on the z -axis. Since in the case of the double surface, such pairs are connected with the cusps of the geodesic hypocycloid from which the straight lines of the surface spring, we may expect in the case of the conjugate surface to find them connected with points on the z -axis from which spring the geodesics. It is to be expected that the relation of tangency between the straight lines and the hypocycloid will be preserved between the geodesics and the axis of z .

16. *The Infinitely Distant Elements.*—These lie upon the same straight lines as in the case of the double surface, but the multiplicity of the real line in the xy -plane is doubled.

17. *Class and Order of the Surface.*—These are fourteen and fifty-six respectively.

18. *Curves of Intersection in the xy -plane.*—These are determined by the equations

$$(A) \lambda^6 + 1 = 0, \quad (B) \sin 3\varphi = 0.$$

Under condition (A)

$$\lambda^4 - \lambda^{-4} = \lambda^2 - \lambda^{-2} = \lambda^2 + \lambda^4.$$

By substituting

$$\lambda = \cos \frac{2g+1}{6} \pi + i \sin \frac{2g+1}{6} \pi, \quad (g \text{ an integer})$$

and reducing, $\lambda^2 + \lambda^4$ may be thrown into the form

$$i c_g = 2i \sin \frac{2g+1}{3} \pi, \quad (g = 0, 1, 2).$$

Putting $\varphi = \frac{\pi - \psi}{2}$, the equations of the section become

$$(40) \quad \begin{aligned} x &= i c_g (-1/4 \sin 2\psi + 1/2 \sin \psi) \\ y &= i c_g (1/4 \cos 2\psi - 1/2 \cos \psi). \end{aligned}$$

These represent three imaginary hypocycloids with vertices on the y -axis as initial points. Of these curves the origin is one, appearing as a limiting case. It corresponds to the value $\lambda^2 = -1$. For this $x = y = z = 0$ irrespective of the value of φ . The origin accordingly is a conical point upon the surface. It appears upon every φ curve, but in general as an isolated point. It is of infinite multiplicity, but since in this case it appears as a hypocycloid it may be regarded as of the same degree as the other two.

These may be shown by means of the relations $uv = -\omega$, $uv = -\omega^2$ to be of degree eight and multiplicity one.

Under condition (B).

$$\frac{\sin 4\varphi}{\sin 2\varphi} = -\frac{\cos 4\varphi}{\cos 2\varphi} = -1$$

and equations (38) reduce to

$$(41) \quad \frac{x}{\sin 2\varphi} = \frac{y}{\cos 2\varphi} = -\frac{1}{4\lambda^4} (\lambda^8 + 2\lambda^6 - 2\lambda^2 - 1) \\ z = 0.$$

These are the equations of three straight lines of multiplicity eight. From the equation

$$(42) \quad \lambda^8 + 2\lambda^6 + 4 \left(\frac{y}{\cos 2\varphi} \right) \lambda^4 - \lambda^2 - 1 = 0$$

it is clear that each line throughout its course is connected with two real nappes of the surface.

The degree of the surface may be estimated from the degrees of the curves of intersection. These are in the xy -plane

3 finite straight lines of multiplicity 8.....	24
1 straight line at infinity of multiplicity 8.....	8
2 imaginary hypocycloids of multiplicity 1, degree 8....	16
The origin.....	8
	56

19. *Curves of Intersection in the yz -Plane.*—These satisfy the conditions.

$$(A) \lambda^4 - 1 = 0, (B) \sin 2\varphi = 0, (C) \cos 2\varphi = \frac{\lambda^2}{\lambda^4 + 1}.$$

Condition (A) breaks up into two; (a) $\lambda^2 = -1$, which gives the origin and (b) $\lambda^2 = 1$. The latter leads to the equations of the z -axis.

$$(43) \quad \begin{aligned} x &= y = 0 \\ z &= -\frac{4}{3} \sin 3\varphi. \end{aligned}$$

Between the points $z = \pm 4/3$ the line is the intersection of six real nappes of the surface since each of its points corresponds to six real values of φ . Beyond the limits $\pm 4/3$ it is isolated.

Condition (B) also breaks up into two; (a) $\varphi = 0$, the condition for the y -axis and (b) $\varphi = \pi/2$ from which

$$(44) \quad \begin{aligned} x &= 0 \\ y &= -\frac{\lambda^8 - 2\lambda^6 - 2\lambda^2 - 1}{4\lambda^4} = -\frac{(\lambda^2 - 1)^3(\lambda^2 + 1)}{4\lambda^4} \\ z &= \frac{2}{3} \frac{\lambda^6 + 1}{\lambda^3} = \frac{2}{3} \frac{(\lambda^2 + 1)(\lambda^4 - \lambda^2 + 1)}{\lambda^3} \end{aligned}$$

When $\varphi = \pi/2$, $u = -v$, and $X = 0$. The curve (44) is therefore the geodesic in the yz -plane. It is of multiplicity one and degree eight. It is symmetric with respect to the axes. Aside from the origin which appears as an isolated point, it has on the axis of y two imaginary double points, and on the axis of z , two real points $\pm 4/3$.

On throwing the equation for y into the form

$$(45) \quad \tau^4 - 2\tau^3 + 4y\tau^2 + 2\tau - 1 = 0, (\tau^2 = \lambda)$$

and forming the discriminant

$$\Delta = -y^2(27 + 16y^2)$$

it is found that y has only one real positive root. It follows that to a given value of y correspond only two real values of z . These are equal and of opposite sign. The curve consists of two detached parts symmetrically disposed to the y -axis. The value of the derivative shows that each of these resembles a semicubical parabola with a cusp on the z -axis at $\pm 4/3$, or $-4/3$.

$$dz = 2 \lambda^{-4} (\lambda^2 - 1) (\lambda^4 + \lambda^2 + 1) d\lambda, \quad dy = -\lambda^{-5} (\lambda^2 - 1)^2 (\lambda^4 + \lambda^2 + 1) d\lambda$$

$$\frac{dz}{dy} = -\frac{2\lambda}{\lambda^2 - 1}.$$

The curve has four imaginary cusps corresponding to the roots of $\lambda^4 + \lambda^2 + 1 = 0$. It becomes parallel to the axis of y at infinity.

From condition (C) result the equations

$$\begin{aligned} x &= 0 \\ (46) \quad y &= \frac{(\lambda^2 - 1)^3 (\lambda^2 + 1)^3}{4 \lambda^4 (\lambda^4 + 1)} \\ z &= -\frac{\sqrt{-2}}{3} \frac{(\lambda^2 + 1)^3 (\lambda^4 - \lambda^2 + 1)^{3/2}}{\lambda^3 (\lambda^4 + 1)^{3/2}} \end{aligned}$$

The curve is double and symmetric with respect to the axes. It is of degree fourteen. On writing the equation for y in the form

$$\tau^3 - (3 + 4y) \tau^2 + (3 - 4y) \tau - 1 = 0, \quad (\tau = \lambda^4)$$

and forming the discriminant

$$\Delta = -64/27 y^2 (4y^2 - 9)$$

it is seen that the twelve values of λ corresponding to a given value of y are imaginary, with the exception of two that are equal and of opposite sign.

The curve resembles the geodesic. It cuts the axes in the same points; consists of two non-intersecting branches symmetric to the axis of y ; has cusps on the z -axis at $\pm 4/3$ and becomes parallel to the y -axis at infinity. It differs from the geodesic in the position of its imaginary cusps, determined from the common factors of dz and dy

$$\begin{aligned} &\frac{(\lambda^2 + 1)^2 (\lambda^2 - 1) (\lambda^8 + 4 \lambda^4 + 1)}{\lambda^4 (\lambda^4 + 1)^3} \\ \frac{dz}{dy} &= \sqrt{-2} \frac{\lambda}{\lambda^2 - 1} \left(\frac{\lambda^4 - \lambda^2 + 1}{\lambda^4 + 1} \right)^{1/2} \end{aligned}$$

There are in the yz -plane :

1	straight line (z -axis) of multiplicity 6.....	6
1	" " (y -axis) " ".....	8
1	geodesic of degree 8 " ".....	8
1	curve " " 14 " ".....	28
	Origin	?

In this case the origin enters through a condition similar in form to that for the axis of z . If it is allowable to regard it here as of the same degree as the z -axis it brings the sum of the degrees of the curves of intersection to the desired figure, fifty-six.

20. *Symmetry of the Surfaces.*—Since the curves in the yz -plane are symmetric with respect to the axes, the three planes of the geodesics can yield but one type of section. This suggests that the surface possesses a second set of planes of symmetry. On substituting for u and v in equations (37), the values $-u, -v; -v, -u; -u^{-1}, -v^{-1}$, it is found that the surface is symmetrical with respect to all of the co-ordinate planes and consequently to the axes. The xz -plane and x -axis, then, belong to a second set of planes and axes of symmetry not discoverable through the theorems of Schwarz.

21. *Curves of Intersection in the xz -Plane.*—The conditions to be satisfied are

$$(A) \lambda^4 - 1 = 0, \quad (B) \frac{\cos 4\varphi}{\cos 2\varphi} = -\frac{2\lambda^2}{\lambda^4 + 1}$$

As before, (A) gives the origin and the z -axis. (B) leads to a curve with the equations

$$(47) \quad x = \frac{(\lambda^4 - 1)(\lambda^8 + \lambda^4 + 1 \pm \lambda^2 \sqrt{2\lambda^8 + 5\lambda^4 + 2})^{1/2} (-3\lambda^2 \pm \sqrt{2\lambda^8 + 5\lambda^4 + 2})}{4\sqrt{2}\lambda^4(\lambda^4 + 1)}$$

$$z = -\frac{(\lambda^6 + 1)(2\lambda^4 + \lambda^2 + 2 + \sqrt{2\lambda^8 + 5\lambda^4 + 2})^{1/2} (\lambda^4 - \lambda^2 + 1 \pm \sqrt{2\lambda^8 + 5\lambda^4 + 2})}{3\lambda^3(\lambda^4 + 1)^{3/2}}$$

For convenience of reference these may be expressed in the form

$$(48) \quad x = \frac{(\lambda^4 - 1) A^{1/2} B}{4\sqrt{2}\lambda^4(\lambda^4 + 1)}$$

$$z = -\frac{(\lambda^6 + 1) C^{1/2} D}{3\lambda^3(\lambda^4 + 1)^{3/2}}$$

From the derivation of equations (47), it is clear that

$$(49) \quad \frac{A^{1/2}}{\sqrt{2}(\lambda^4 + 1)} = \sin 2\varphi, \quad \frac{C^{1/2}}{4(\lambda^4 + 1)} = \sin \varphi.$$

The radical $M \equiv \sqrt{2\lambda^8 + 5\lambda^4 + 2}$ is real for every real value of λ and is less in absolute value than $(2\lambda^4 + \lambda^2 + 2)$ or than $\lambda^{-2}(\lambda^8 + \lambda^4 + 1)$; x and z are therefore real for all real values of λ . Since z is imaginary for all imaginary values of λ , except those for which it is zero or infinite, the curve has no isolated branches.

On account of the possible combinations of the radicals involved, to a given value of λ correspond eight points. These fall into two groups of four each, according as M is taken with the upper or the lower signs in (47). The four points of a group are symmetrically situated in the four quadrants; $\pm \lambda_0, \pm 1/\lambda_0$ give the same points. Apparently the curve is of multiplicity four, but the consideration of equations (49) shows that such is not the case. When λ is fixed, the possible combinations of the signs of $A^{1/2}$ and $C^{1/2}$ indicate four related values of φ , namely, $\varphi, \pi - \varphi, \pi + \varphi, 2\pi - \varphi$. The change of λ to $-\lambda$ gives rise to no new values of u and v . The negative values of λ are therefore not to be counted in estimating the multiplicity of the curve.

To discover the nature of the curve, it suffices to consider the change in the co-ordinates as λ ranges from 1 to ∞ . When $\lambda = 1, x = 0$ and $z = \pm 4/3$. An examination of the factors entering into the numerators and denominators of x and z shows that they all increase in absolute value with λ ($\lambda > 1$), whether M be taken with the upper or the lower signs throughout. It follows that the curve consists of two real branches in each quadrant, taking their rise on the axis of z at $\pm 4/3$ or $-\pm 4/3$ and proceeding to infinity.

22. Curves Corresponding to the Parallels and Meridians.

The λ Curves.—These are of the eighth degree, with the exception of the axis of z , which is of multiplicity six. To a curve λ on the double surface correspond two on the conjugate, furnished by λ and $1/\lambda$. Each of these curves is symmetric with respect to the xy - and yz -planes, and the two are symmetric to each other with respect to the xz -plane. They accordingly intersect in this plane on the double curve (47).

A given curve of the system cuts the xy -plane in three double points one on each of the straight lines $x = 0, x = \mp \sqrt{3}y$; the yz -plane, in a double point on the y -axis, two double points on the double curve and two ordinary points on the geodesic; and the xz -plane, in eight ordinary points, two in each quadrant, one on each branch of the double curve.

The φ Curves.—These are of degree eight. When φ is fixed, the substitutions $\pm \lambda, \pm 1/\lambda$ show the curve to be symmetric to the xy -plane and the origin. Since, except in the case of the

straight lines, the intersections of the curves with the xy -plane are imaginary or isolated, each curve consists of two non-intersecting branches. Curves φ and $\pi - \varphi$ are symmetric to each other with respect to the xz - and yz -planes and intersect on the double curves in those planes. Through each point on the z -axis between $\pm 4/3$ pass six φ -curves.

The direction cosines of the tangent are

$$\frac{(\lambda^8 + 1) \sin 4\varphi - \lambda^2 (\lambda^5 + 1) \sin 2\varphi}{P_{1/2}},$$

$$\frac{(\lambda^8 + 1) \cos 4\varphi + \lambda^2 (\lambda^5 + 1) \cos 2\varphi}{P_{1/2}}, \quad \frac{-2\lambda (\lambda^4 - 1) \sin 3\varphi}{P_{1/2}}$$

where

$$P_{1/2} =$$

$$\sqrt{(\lambda^8 + 1)^2 + \lambda^4 (\lambda^5 + 1)^2 + 2\lambda^2 (\lambda^5 + 1) (\lambda^8 + 1) \cos 6\varphi + 4\lambda^2 (\lambda^4 - 1)^2 \sin 3\varphi}$$

These values show that the φ -curves, with the exception of the geodesics, cut the axis of z orthogonally, and all become parallel to the xy -plane at infinity.

23. *Model of the Surface.*—The following values multiplied throughout by two furnished the data for a skeleton model of the surface, made on the same plan as that of the double surface.

GEODESIC IN yz -PLANE.			DOUBLE CURVE IN yz -PLANE.		
λ	y	z	λ	y	z
1	0	1.34	1	0	1.34
1.2	— .03	1.54	1.14	.02	1.44
1.4	— .17	2.07	1.299	.14	1.76
1.6	— .51	2.89	1.48	.49	2.32
1.686	— .75	3.34	1.686	1.22	3.19

DOUBLE CURVE IN xz -PLANE.

BRANCH I. (M taken with upper signs.)			BRANCH II. (M taken with lower signs.)		
λ	x	z	λ	x	z
1	0	1.34	1	0	1.34
1.14	.01	1.44	1.14	.12	1.40
1.299	.09	1.76	1.299	.49	1.60
1.48	.32	2.36	1.48	1.16	1.95
1.686	.84	3.18	1.686	2.26	2.36

INTERSECTIONS OF CERTAIN λ - AND φ -CURVES.

λ		$\varphi = 0$	$\varphi = \frac{\pi}{24}$	$\varphi = \frac{2\pi}{24}$	$\varphi = \frac{3\pi}{24}$	$\varphi = \frac{4\pi}{24}$
$\varepsilon^0 = 1$	x	0	0	0	0	0
	y	0	0	0	0	0
	z	0	— .51	— .94	— 1.23	— 1.33
$\varepsilon^{\frac{\pi}{24}} = 1.14$	x	0	.07	.10	.09	.01
	y	— .54	— .49	— .37	— .19	0
	z	0	— .55	— 1.02	— 1.33	— 1.44
$\varepsilon^{\frac{2\pi}{24}} = 1.299$	x	0	.17	.27	.24	.07
	y	— 1.12	— 1.07	— .79	— .39	.04
	z	0	— .68	— 1.25	— 1.63	— 1.77
$\varepsilon^{\frac{3\pi}{24}} = 1.48$	x	0	.35	.56	.53	.24
	y	— 2.02	— 1.83	— 1.32	— .61	.14
	z	0	— .91	— 1.68	— 2.19	— 2.37
$\varepsilon^{\frac{4\pi}{24}} = 1.686$	x	0	.68	1.10	1.11	.65
	y	— 3.24	— 2.93	— 2.08	— .88	.37
	z	0	— 1.28	— 2.36	— 3.09	— 3.34

My sincere thanks are due to Professor F. D. Sherman, of Columbia University, who from the wire model made the drawing of the surface reproduced in Plate II.

III. PROPERTIES OF THE PRIMITIVE SURFACES IN GENERAL.

When the methods used in the preceding pages are applied to the discussion of the primitive surfaces double and conjugate for a general value of μ , results are obtained that are presented below in tabular form. By reference to page 4 the results due to the paper already cited may be distinguished. In treating the conjugate surfaces, only those points in which they differ from the corresponding double surfaces will be noted.

Primitive Surfaces.

Double

 μ either odd or even.

Conjugate

1. Properties of the minimal curve :
Order, $2\mu + 4$; class, $2\mu + 2$; rank,
 $2\mu + 4$.

Multiplicity of circle at infinity on
tangential surface is 1.

Number of coincident points at
infinity, $2\mu + 4$.

Plane of osculation at infinitely
distant points falls into plane at
infinity, and tangent line becomes
tangent to imaginary circle.

Stationary points are furnished by
the roots of $u^{2\mu} + z = (-1)^{\mu} + 1$.

2. Class of the Surface, $2\mu + 3$.

Order of the Surface, $(2\mu + 3)(\mu + 2)$.

3. Infinitely distant elements lie
on three straight lines of multiplicity
 $(\mu + 2)$ with equations.

$$z = 0 \quad \tau = 0$$

$$x + i y = 0 \quad \tau = 0$$

$$x - i y = 0 \quad \tau = 0$$

4. All plane lines of curvature
on the surface are geodesics.

No asymptotic lines exist corre-
sponding to small circles on the
sphere.

The surface possesses $(\mu + 1)$
geodesics lying in planes passing
through the axis of z . To these
planes the xz -plane always belongs.

The surface possesses $(\mu + 1)$
straight lines lying in the xy -plane
and passing through the origin.

5. There are no vertical planes of
symmetry, nor axes of symmetry
that lie in the xy -plane, aside from
those determined by the geodesics
and straight lines mentioned above.

Class, $2(2\mu + 3)$.

Order, $2(2\mu + 3)(\mu + 2)$.

The real line is of multiplicity
 $2(\mu + 2)$.

To these planes the xz -plane does
not always belong.

Among the $(\mu + 1)$ lines, the axis
of y is always included.

5. There are $(\mu + 1)$ additional
planes and $(\mu + 1)$ axes of symmetry,
bisecting the angles between the
planes and lines of the first set.
The planes of the second set do not
contain geodesics. The lines do
not lie on the surface.

Also the xy -plane and z -axis are
respectively a plane and axis of
symmetry.

 μ even.

1. The surface possesses a geo-
desic in the xy -plane. The curve
is a hypocycloid with $(\mu + 1)$ cusps.
It is cut by each straight line of the
surface in a cusp, an ordinary point
and $\frac{\mu-2}{2}$ double points.

The xy -plane is a plane of sym-
metry containing a geodesic.

1. The z -axis between the points
 $\pm 4/\mu + 1$ is connected with $2(\mu + 1)$
real nappes of the surface. Outside
of these limits it is isolated.

The xy -plane is a plane of sym-
metry, not containing a geodesic.

2. The z -axis does not lie on the surface, and is not an axis of symmetry.

3. The $(\mu + 1)$ vertical planes of symmetry pass through the $(\mu + 1)$ straight lines and both pass through the cusps of the hypocycloid. To the lines always belongs the axis of x .

4. Pairs of conjugate stationary points of the minimal curve are symmetric to the xy -plane, and the middle points of lines joining such pairs are the cusps of the geodesic hypocycloid.

5. The curves of intersection in the xy -plane are

$(\mu + 1)$ finite straight lines of multiplicity $(\mu + 2)$	$(\mu + 1)(\mu + 2)$
1 line at infinity of multiplicity $(\mu + 2)$	$\mu + 2$
$\mu/2$ isolated hypocycloids of multiplicity 2, degree $(\mu + 2)$..	$\mu(\mu + 2)$
1 geodesic hypocycloid of multiplicity 1, degree $(\mu + 2)$..	$\mu + 2$
	$(2\mu + 3)(\mu + 2)$

All of the hypocycloids may be derived from any one of the number by factors of proportionality.

The $(\mu + 1)$ straight lines are tangent to the geodesic at the cusps and from these points outward to infinity are each connected with two real nappes of the surface.

6. The sections by the vertical planes of symmetry are of two types differing in the position of corresponding curves with respect to the axis of z .

The curves of intersection in the xz -plane are:

- 1 straight line (x -axis) of multiplicity $(\mu + 2)$.
- 1 geodesic of multiplicity 1, and degree $(\mu + 2)$.
- 1 multiple curve.

The geodesic resembles an ordinary parabola. It cuts the x -axis orthogonally at the simple point on the geodesic hypocycloid. It is symmetric to the x -axis and cuts the z -axis in two real points. Through these pass all the $(\mu + 1)$ vertical geodesics.

2. The z -axis lies on the surface, and is an axis of symmetry.

3. The planes of the geodesics and the straight lines of the surface are still united in position; but the hypocycloids are imaginary. To the planes always belongs the yz -plane.

4. Pairs of conjugate stationary points are symmetric to the axis of z and furnish the points $\pm 4/\mu + 1$.

5. The curves of intersection in the xy -plane are:

$(\mu + 1)$ finite straight lines of multiplicity $2(\mu + 2)$	$2(\mu + 1)(\mu + 2)$
1 straight line at infinity of multiplicity $2(\mu + 2)$	$2(\mu + 2)$
$\mu + 1$ imaginary hypocycloids (including the origin) of multiplicity 1, and degree $2(\mu + 2)$	$2(\mu + 1)(\mu + 2)$
	$2(2\mu + 3)(\mu + 2)$

The $(\mu + 1)$ straight lines throughout their course are each connected with two real nappes of the surface.

6. The sections are of two types according as the cutting plane does or does not contain a geodesic. The yz -plane furnishes a section of the first type; the xz -plane, of the second,

The curves in the yz -plane are:

- 1 conical point, the origin,
- 1 straight line (the axis of z) of multiplicity, $2(\mu + 1)$.
- 1 geodesic of multiplicity 1, and degree $2(\mu + 2)$.
- 1 multiple curve.

The geodesic is symmetric to the axes. It has two cusps on the axis of z at $\pm 4/\mu + 1$, and cuts the y -axis in imaginary or isolated points. It consists of two branches, each resembling a semi-cubical parabola.

The curves in the xz -plane are

- 1 conical point, the origin.

7. The λ curves are of degree $2(\mu + 2)$ with the exception of the geodesic hypocycloid and the real line at infinity. These are of degree $(\mu + 2)$. The curves are symmetric to all the planes and axes of symmetry and cut the xy -plane in $(\mu + 1)$ double points, one on each of the straight lines.

8. The φ curves are of degree $(\mu + 2)$. They are symmetric to the xy -plane, which they cut orthogonally on the geodesic. They become parallel to the xy -plane at infinity.

1 straight line (the z -axis) of multiplicity $2(\mu + 1)$.

1 multiple curve.

7. The λ curves are of degree $2(\mu + 2)$ with the exception of the axis of z . They are symmetric to the planes of the geodesics and the straight lines of the surface. Two, λ and $1/\lambda$, are symmetrically placed with reference to the remaining planes of symmetry and intersect on multiple curves in these planes. Both curves correspond to the one curve λ on the double surface.

8. The φ curves are of degree $2(\mu + 2)$. They are symmetric to the origin and the xy -plane. Each consists of two non-intersecting branches symmetrically placed with reference to the xy -plane and cutting the axis of z orthogonally between $\pm 4/\mu + 1$. $2(\mu + 1)$ of the curves pass through each non-isolated point of the axis of z , and all become parallel to the xy -plane at infinity.

μ odd

1. The axis of z between the points $\pm 4/\mu + 1$ is connected with $(\mu + 1)$ real nappes of the surface. Outside of these limits it is isolated.

2. The vertical planes of the geodesics always include the xz - and yz -planes.

The straight lines in the xy -plane bisect the angles between the planes of the geodesics.

3. Pairs of conjugate stationary points of the minimal curve are symmetric to the z axis, and furnish the points $z = \pm 4/\mu + 1$, from which the geodesics spring.

4. The curves of intersection in the xy -plane are:

$(\mu + 1)$ finite
straight lines
of multiplicity
 $(\mu + 2) \dots (\mu + 1)(\mu + 2)$

1 straight line at
infinity..... $\mu + 2$

$\frac{\mu + 1}{2}$ imaginary
hypocycloids
of multiplicity
1, and degree
 $2(\mu + 2) \dots \frac{(\mu + 1)(\mu + 2)}{(2\mu + 3)(\mu + 2)}$

1. The surface possesses a geodesic hypocycloid in the xy -plane. This has $2(\mu + 1)$ cusps. It is cut by each straight line of the surface in two cusps and $(\mu - 1)$ double points.

2. The vertical planes of the geodesics always bisect the angles between the $(\mu + 1)$ straight lines.

These lines always include the x - and y -axes.

3. Pairs of conjugate stationary points are symmetric to the xy -plane and furnish the cusps of the geodesic hypocycloid.

4. Curves in the xy -plane:

$(\mu + 1)$ finite
straight lines
of multiplicity
 $2(\mu + 2) \dots 2(\mu + 1)(\mu + 2)$

1 straight line
at infinity of
multiplicity
 $2(\mu + 2) \dots 2(\mu + 2)$

1 geodesic hypo-
cycloid of de-
gree $2(\mu + 2) \dots 2(\mu + 2)$

$\frac{\mu - 1}{2}$ isolated
hypocycloids
of multiplicity
2, degree
 $2(\mu + 2) \dots 2(\mu - 1)(\mu + 2)$

The straight lines are connected with real parts of the surface throughout their course. One real nappe passes through each.

5. The sections by the vertical planes of symmetry are of two types differing in the position of corresponding curves with respect to the x -axis.

The curves of intersection in the yz -plane are:

1 straight line (z -axis) of multiplicity $(\mu + 2)$.

1 geodesic of degree $(\mu + 2)$.

1 multiple curve.

The geodesic is symmetric to the axis of z on which it has a cusp at $-4/\mu + 1$. Its intersections with the x -axis are imaginary. It resembles a semi-cubical parabola.

6. The λ curves are of degree $2(\mu + 2)$ with the exception of the z -axis, and the real line at infinity. The curves are symmetric to the planes and lines of symmetry and cut the xy -plane in $2(\mu + 1)$ ordinary points, two symmetrically placed with reference to the origin on each of the $(\mu + 1)$ straight lines.

7. The φ curves are of degree $(\mu + 2)$. They are symmetric to the z -axis, and with the exception of the $(\mu + 1)$ straight lines have no real points in the xy -plane. $\mu + 1$ of the curves pass through each non-isolated point of the z -axis. All become parallel to the xy -plane, at infinity.

1 hypocycloid
(the origin)
of multiplicity
1, degree
 $2(\mu + 2)$ $\frac{2(\mu + 2)}{2(2\mu + 3)(\mu + 2)}$

The straight lines are tangent to the geodesic hypocycloid at the cusps and proceed outward from these points in connection with two real nappes.

5. The sections are of two types according as the planes do or do not contain geodesics. None of the planes of the geodesics coincides with a co-ordinate plane. But by rotation of the surface one may be brought into coincidence with the yz -plane. The curves of intersection are found to be:

1 conical point, the origin.

1 geodesic of degree $2(\mu + 2)$.

1 multiple curve.

The geodesic consists of two branches, each of the same type as the vertical geodesics of the double surfaces μ even. These branches intersect on the z -axis.

The curves of intersection in the yz -plane are:

1 conical point, the origin.

1 straight line (y -axis) of multiplicity $2(\mu + 2)$.

1 multiple curve.

6. The λ curves are of degree $2(\mu + 2)$. Each is symmetric with respect to the z -axis and to the vertical planes of the geodesics. Curves λ and $1/\lambda$ are symmetric to each other with respect to the remaining planes of symmetry, and give by their intersection, multiple curves in those planes. Both correspond to a curve λ on the double surface.

7. The φ curves are of degree $2(\mu + 2)$. They are symmetric to the origin and consist of two non-intersecting branches symmetrically disposed to the z -axis. They cut the geodesic hypocycloid orthogonally and become parallel to the xy -plane at infinity.

PLATE I.

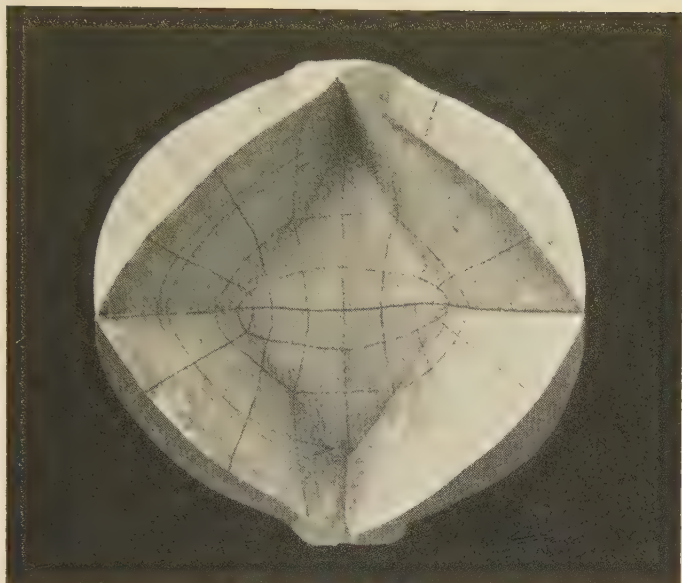


FIG. 1.

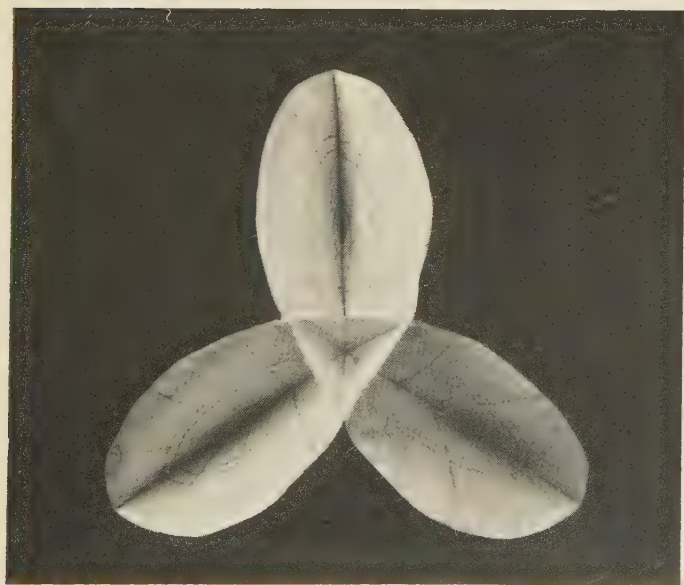


FIG. 2.

PLATE II.

